Wavelet-Galerkin Solutions of One and Two Dimensional Partial Differential Equations

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ABSTRACT

In recent years wavelets have received much attention because of its comprehensive mathematical power and good application potential in many interesting branches of science and technology. The advantage of wavelet techniques over finite difference or element method is well known. This paper offers wavelet based Galerkin methods for solving partial differential equations and provides some examples as test problems.

1. INTRODUCTION

Since the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and multiresolution analysis based fast wavelet transform algorithm by Beylkin (1991), wavelet based approximation of ordinary and partial differential equations gained momentum in attractive way. Wavelets have the capability of representing the solutions at different levels of resolutions, which make them particularly useful for developing hierarchical solutions to engineering problems. Among the approximations, wavelet-Galerkin technique is the most frequently used scheme these days. Daubechies wavelets as bases in a Galerkin method to solve differential equations require a computational domain of simple shape. This has become possible due to the remarkable work by Amartunga et al. (1993, 1994 & 1996) [1-3], Latto et al. (1992) [5], Xu et al. (1994) [10] and Williams et al. (1993 & 1994) [8-9]. Yet there is difficulty in dealing with boundary conditions. So far problems with periodic boundary conditions or periodic distribution have been dealt successfully.

Advantage of wavelet-Galerkin method over finite difference or element method has lead to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Although the wavelet method provided an efficient alternative technique for solving partial differential equations [PDEs] numerically, it is not as easy to implement as the traditional finite-difference method. The reason is that the use of the wavelet-Galerkin method to solve PDEs leads to the problem of computing integrals whose integrands involves products of compactly supported wavelets and their derivatives. These integrals are evaluated using what is known as the ‘connection coefficient method’. Notice that with increasing resolution (for Daub6) accuracy deteriorates, since condition number increases. But for higher order wavelets, condition number is consistently lower. Moreover, the condition number also depends on the order of derivatives, \( \Omega^{0,d} \) increases with increase in derivatives \( d \).

Wavelet-Galerkin methods such as Amaratunga et al. method, fictitious boundary approach, capacitance matrix method, difference wavelet-Galerkin difference method and wavelet Taylor Galerkin approach for PDEs are well known due to their own advantages. For wavelet-Galerkin technique for ordinary differential equations refer to Mishra et al. [6-7].

2. 2D MULTIGRID ANALYSIS

2D Multiresolution Analysis [MRA] (Christov [4]) can be constructed by taking tensor product space \( F \otimes G = \{ f(x)g(y) : f \in F, g \in G \} \) of 1D ones. Let \( L^2(\mathbb{R}^2) \) be the space of two-dimensional square integrable function. Given an MRA \( \{ V_j \} \) with scaling function \( \varphi_{n,k} \) and corresponding wavelet space \( \{ W_j \} \) with wavelet \( \psi_{n,k} \), define

\[
\varphi_{n,(k_1,k_2)}(x,y) = \varphi_{n,k_1}(x) \otimes \varphi_{n,k_2}(y)
\]

\[
\psi^{1}_{n,(k_1,k_2)}(x,y) = \psi_{n,k_1}(x) \otimes \varphi_{n,k_2}(y)
\]

\[
\psi^{2}_{n,(k_1,k_2)}(x,y) = \varphi_{n,k_1}(x) \otimes \varphi_{n,k_2}(y)
\]

\[
\psi^{3}_{n,(k_1,k_2)}(x,y) = \psi_{n,k_1}(x) \otimes \psi_{n,k_2}(y)
\]

For any function \( f(x,y) \subseteq L^2(\mathbb{R}^2) \), projection of \( f \) onto scaling space \( V_m \) at resolution \( m \) may be defined by

\[
P_m f(x,y) = \sum_k \sum_i c_{k,i} \varphi_{m,k}(x) \varphi_{m,i}(y)
\]
3. 2D WAVELET-GALERKIN TECHNIQUE

Consider the following problem:

\[ L[u(x,y)] = 0 \] on the region \( S(x,y) \)

With boundary conditions \( D(u) = 0 \) on the boundary \( \tau \) of \( S \).

Assume that \( u(x,y) \) can be represented accurately by a set of analytic function \( \{g_i(x,y)\}_{i=1}^{N} \) such that

\[ u(x,y) \approx u_0(x,y) + \sum_{i=1}^{N} a_i g_i(x,y) = u_\alpha(x,y) \]

\( u_0 \) is so chosen as to satisfy the initial conditions. For approximation \( u_\alpha \) to be good, the residual \( R \) of

\[ L[u_0(x,y)] + L[\sum_{i=1}^{N} a_i g_i(x,y)] = R(a_1, ..., a_N, x, y). \]

must reduce to minimum, for that we use the simplest Galerkin Method, namely Ritz-Raleigh. This method minimizes residual to the effect that

\[ \langle R(a_1, ..., a_N, x, y), g_i(x,y) \rangle_{L^2} = 0, i = 1,2, ..., N \]

which in turn gives

\[ \sum_{i=1}^{N} a_i \langle L[g_i(x,y)], g_i(x,y) \rangle + \langle L[u_\alpha(x,y)], g_i(x,y) \rangle = 0 \]

Next step is to find \( \{a_i\}_{i=1}^{N} \) form the matrix equation formation (1)

\[ GA = U, \quad G_{ij} = L[g_i, g_j], \quad a = [a_i], \quad U_j = L[u_\alpha, g_j]. \]

Let \( \{V_j\}_{j \in Z} \) be an MRA with scaling function \( \varphi_k(x) = \sum \varphi_k \varphi(2x - k) \).

\( \{\varphi_{jk} = 2^{j/2}\varphi(2^{j}x - k), k, j \in Z\} \) acts as orthonormal basis for \( V_j \). At each approximation level \( j \), orthogonal projection of \( u(x,y) \) onto \( V_j \) is taken in the manner (fit \( y \) fixed)

\[ u(x,y) \cong \sum_k a_{j,k} \varphi_{jk}(x), \quad a_{j,k} = \langle u(x,y), \varphi_{jk} \rangle \]

\[ u(x,y) \cong \sum_k a_{j,k} \varphi_{jk}(x), \quad a_{j,k} = \langle u(x,y), \varphi_{jk} \rangle \]

For some \( J, V_j \) will capture all details of the original function. Select \( k \in \{0,1, ..., 2^n - 1\} \). Substituting (1) and forcing the condition (2), we find

\[ \langle L[y_j](u(x,y)) + Lu_0(x,y), \varphi_{jk}(x) \rangle = 0. \]

Clearly this is 1D problem and cannot be applied to 2D. Let us assume \( a_{j,k} \) function of \( y \) and for each \( y_j \), we can solve the system as for 1D problem. Ultimately we obtain

\[ BA = R, B_1 = \langle L[R_1 \varphi_{j,k}(x), \varphi_{j,m}], A_i = a_{j,k}(y), R_1 = \langle L[u_0(x,y), \varphi_{j,k}(x) \rangle \]

Solving (5), we find for each \( y \) the coefficients \( a_{j,k}(y) \) and thus the solution to (1).

For details refer to Christov [4].

4. WAVELET METHODS FOR PDES

4.1 Wavelet-Galerkin Solution of the Periodic Problem [3]

Consider the two dimensional Poission’s equation

\[ u_{xx} + u_{yy} = f, \]

Where \( u = u(x,y), f = f(x,y) \) are periodic in \( x, y \) of period \( d_x, d_y \in Z \).

Let the approximate solution \( u(x,y) \) at scale \( m \) be

\[ u(x,y) = \sum_k \sum_l c_{kl} \varphi(2^m x - k) \varphi(2^m y - l), k, l \in Z \]

Where \( c_{kl} \) are periodic wavelet coefficients of \( u \).

Put \( X = 2^m x, Y = 2^m y \) so that

\[ U(X,Y) = u(x,y) = \sum_k \sum_l c_{kl} \varphi(x - k) \varphi(y - l), c_{kl} = c_{kl} 2^{m/2} \]

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\[ U(X, Y) \] and also \( C_{k,l} \) are periodic in \( X, Y \) with periods \( n_x = 2^m d_x, n_y = 2^m d_y \). Let us discretize \( U(X, Y) \) at all dyadic points
\[ x, y = 2^{-m} x, 2^{-m} y, X, Y \in Z \]
\[ U_{ij} = \sum_{k} \sum_{l} C_{k,l} \phi_{k-x} \phi_{l-y} = \sum_{k} \sum_{l} C_{k,l} \phi_{k} \phi_{l}, \quad i = 0, 1, 2, ..., n_x - 1, \]
\[ j = 0, 1, 2, ..., n_y - 1 \]

The matrix representation is
\[ U = k_{\varphi_x} * k_{\varphi_y} * C, \tag{9} \]
where \( k_{\varphi_x}, k_{\varphi_y} \) are the convolution kernels, i.e. the first column of the scaling function matrices and \( C \) is the wavelet coefficient matrix.

Similarly, RHS of (6) can be expressed for
\[ f(x, y) = \sum_{k} \sum_{l} d_{k,l} 2^{-m/2} \varphi(2^m x - k) 2^{-m/2} \varphi(2^m y - l), k, l \in Z \]

as
\[ F(X, Y) = f(x, y) = \sum_{k} \sum_{l} D_{k,l} \varphi(X - k) \varphi(Y - l), D_{k,l} = 2^{m/2} d_{k,l}, \tag{10} \]

takes as
\[ F = k_{\varphi_x} * k_{\varphi_y} * D. \tag{11} \]

Substitute the expansion of \( u(x, y) \) and \( f(x, y) \) into the given differential equation (6) and then take inner product on both sides with \( \varphi(X - p), \varphi(Y - q) \).

Use
\[ \Omega_{k} = \int \varphi^n(y - k) \varphi(y - j) dy \quad \text{and} \quad \delta_{k} = \int \varphi(y - k) \varphi(y - j) dy. \]

We obtain, \( k_{\varphi} x C = \frac{1}{2^{2m}} G \).

Taking Fourier Transforms of (9), (10) and (11), we find
\[ \hat{U} = \frac{1}{2^{2m}} \hat{F}. \]
Inverse FT gives the solution \( U \).

4.2 Capacitance Matrix Method and the Boundary Conditions \[ [3] \]
Consider the problem (6), i.e. \( u, f \) are periodic with period \( d_x, d_y \) with Dirichlet’s boundary conditions \( u = u_{\tau}(x, y) \) on the boundary \( \tau \) of region \( S \). If \( f \) is not periodic, it can be made periodic making it zero or extending smoothly outside \( S \). Let \( \psi(x, y) \) be the solution in \( S \) with periodic boundary conditions. The solution \( u(x, y) \) to the differential equation with Dirichlet boundary conditions is obtained by adding another function \( w(x, y) \) such that \( u = v + w \). Since
\[ v_{xx} + v_{yy} = f, \]
\( w \) must satisfy \( w_{xx} + w_{yy} = 0 \) in \( S \).

However, on or outside \( \tau \), \( \nabla^2 w = w_{xx} + w_{yy} \) may take such values as to make \( u \) satisfy the given boundary conditions. The desired effect may be achieved by placing sources (or delta function) along a closed boundary \( \tau_1 \) which encompasses the region \( S \). In other words, \( w \) is given by
\[ u_{xx} + w_{yy} = X_1 \text{in } [0, d_x], [0, d_y]. \]

where
\[ X_1 = X_1(x, y) = \int_{\tau_1} X_0(p, q) \delta(x - p, y - q) d\tau_1 \]
and \( \delta(x, y) \) is the delta function at \( (0, 0) \). So the solution
\[ w(x, y) = G(x, y) \times X_1(x, y) = \int_{\tau_1} X_0(p, q) G(x - p, y - q) d\tau_1 \]
\[ (p, q) \in \tau_1. \tag{12} \]

Discretize (12) at the points \( (p_j, q_j), j = 1, 2, ..., n_{\tau} \) on \( \tau_1 \), \( n_{\tau} \) stands for number of points on \( \tau \).
\[ w(x, y) = \sum_j X_j G(x - p_j, y - q_j) \]
\[ X_j = X_0(p_j, q_j). \tag{13} \]

Similarly considering the mesh points \( (x_i, y_i), i = 1, 2, ..., n_{\tau} \) on \( \tau \). The discretized solution (13) takes the form
\[ w_i \equiv w(x_i, y_i) = \sum_j X_j G(x_i - p_j, y_i - q_j) \]
Solving the matrix formation,
\[ w = Gx, \quad w = [w_i], \quad G = [G_{ij}] = \
G(x_i - p_j, y_i - q_j), X = [X_i] \]
yields \( X_j \) which when substituted in (13) gives \( w \).

### 4.3 Wavelet-Galerkin Fictitious Boundary Approach

Consider the equation \( u_{xx} + u_{yy} = f \).

Let approximate solution be

\[ u(x, y) = \sum_k \sum_l c_{kl} \varphi(2^m x - k) \varphi(2^m y - l); \]

\[ = \sum_k \sum_l c_{kl} \varphi(X - k) \varphi(Y - l); \]

\[ c_{kl} \varphi(2^m x = X, \quad 2^m y = Y) \]

Similarly

\[ f(x, y) = \sum_k \sum_l D_{kl} \varphi(X - k) \varphi(Y - l), \quad \text{where} \]

\[ D_{kl} \varphi(2^m x = D_{kl}. \]

Substituting in the given equation and letting inner product with \( \varphi(Y - q) \)

\[ (p, q \in Z) \]
gives

\[ \sum_k \sum_l 2^m c_{kl} \int \varphi(X - k) \varphi(X - p) \varphi(Y - q) dy \]

\[ + \sum_k \sum_l 2^m c_{kl} \int \frac{\partial}{\partial x} (X - k) \varphi(X - p) \varphi(Y - q) dy \]

\[ = \sum_k \sum_l D_{kl} \int \varphi(X - k) \varphi(X - p) \varphi(Y - q) dy. \]

Or

\[ \sum_k C_{k,q} \varphi(X - k) \varphi(Y - l) \varphi(Y - q) dy \]

\[ = \frac{1}{2^m} \sum_{p,l} D_{p,l}. \]

That is,

\[ \sum_k C_{k,q} \Omega_{p-k} + \sum_l C_{p,l} \Omega_{q-l} = \frac{1}{2^m} D_{p,l} \]

where

\[ \Omega_{p-k} = \int \varphi(X - k) \varphi(X - p) \varphi(Y - q) dy \]

\[ \Omega_{q-l} = \int \varphi(Y - l) \varphi(Y - q) dy \]

are the connection coefficients.

Solving (15) will give the coefficients and hence the solution.

### 5. SOLUTIONS OF PDES: TEST PROBLEMS

**Problem 1:** Consider the 1D wave equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]

with initial condition \( u(x, 0) = u_0(x) \) and the boundary condition

\[ u(0, t) = u(1, t). \]

Using 2-term connection coefficients and following the method as in Section 4.3, the comparable solution with exact one

\[ \sin \pi x e^{-t c^2 \pi^2} \]

with

\[ t_{min} = 0, t_{max} = 1, \quad \text{no. of time steps} = 10, \]

\[ u(0, t) = u(1, t), c = 2 \times 10^{-5} \]

at \( N = 7 \) is shown in Fig. 1.
Problem 2: Consider 2D problem

\[ u_{xx} + u_{yy} = 1 \]

with

\[ u(x,y,t) \]

\[ u_x |_{x=0} = u_y |_{y=0} = 0 \; ; \; u |_{x=1} = u |_{y=1} = 0. \]

Fig 2: Solution 2D Problem (Problem 2) with error estimation for \( N = 10, \; j = 7. \)
Above is the 3D graph of solution obtained through wavelet-Galerkin technique and error in comparison to finite difference solution.

6. CONCLUSION
Despite some disadvantages wavelet-Galerkin technique provides efficient solutions for PDEs.

The graph of wavelet-Galerkin solutions for 1D and 2D are shown with error estimates in comparison to known/finite difference solutions. Fairly good solutions are obtained with wavelet-Galerkin technique.

REFERENCES


