Largest Empty Axis-Parallel Rectangular Annulus

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ABSTRACT

In Euclidean plane, a rectangular annulus is the region between parallel rectangles such that the smaller rectangle lies wholly inside the outer rectangle. Given a set \( P \) of \( n \) points in the two dimensional plane, we propose \( O(n^2) \) time and \( O(n) \) space algorithm to identify an axis-parallel largest empty annulus amidst the points of \( P \). We are not aware of any published work on this problem. To the best of our knowledge this is the first \( O(n^2) \) algorithm for recognition largest empty annulus.

Keywords: Geometric Object, Rectangular Annulus, Minimum Enclosing Rectangle; Computational Geometry

1 Introduction

Recognition of an empty geometrical object of maximum area or perimeter among a point set has been studied extensively in computational geometry [15, 5]. In some cases, the geometrical object is orientation-invariant, that is, the region bounded by the object remains same under rotation (for example circle). However, if the enclosing object is orientation-dependent (for example rectangle), then it sometimes become more difficult to compute the optimal orientation over all possible orientations. In this work, we identify a maximum-width empty rectangular annulus amidst a point set \( P \).

A general problem in rectangular domain is to recognize a largest empty rectangle of arbitrary orientation among a point set [1]. Nandy et al. [6] and Mukhopadhyay et al. [12] independently give \( O(n^3) \) algorithm to compute the largest empty rectangle of arbitrary orientation amidst a set of \( n \) points. Note that dropping the restriction of arbitrary orientation and allowing fixed orientation, makes the problem more simpler. The best known algorithm for computing an axis-parallel largest empty rectangle runs in \( O(n \log^2 n) \) time [2]. A related problem identifies the largest axis-parallel rectangle inside a simple polygon in \( O(n \log n) \) time [7].

Another variation in this area identifies an empty convex \( k \)-gon of maximum area or perimeter among a point set [1]. Recently, Nandy et al. [14] identifies an empty orthoconvex axis-parallel polygon of arbitrary shape and maximum area in \( O(n^3) \) time using \( O(n^2) \) space. An axis-parallel polygon is said to be orthoconvex if the intersection of the polygon with any horizontal or vertical line is a line segment. Das et al. [10] proposed \( O(n^3 \log^2 n) \) time and \( O(n^2) \) space algorithm to identify the widest empty 1-corner corridor, improving the time complexity result of Díaz-Báñez et al. [9] by a factor of \( \log n \). The reader is referred to [1, 3, 4, 13] and the references there in for further study and applications.

In Euclidean plane, a circular annulus (annulus) is the region between two concentric circles. Given a set \( P \) of \( n \) points in the Euclidean plane, Díaz-Báñez et al. [8] address the problem of computing an empty annulus \( \mathcal{A} \) of largest width. In this work, the set \( P \) is partitioned so that no point \( p (\in P) \) lies in the interior of the empty annulus \( \mathcal{A} \) and width of the annulus is largest. They first compute the centers of annuli which are locally optimal and then show that the problem can be solved in \( O(n^3 \log n) \) time and \( O(n) \) space. Very recently, Mukherjee et al. [11] studies the problem of computing an arbitrary oriented minimum-width rectangular annulus that encloses a set of \( n \) points. Their algorithm runs in \( O(n^2 \log n) \)
time and \(O(n)\) space. Based on the motivation of this problem, we consider the problem of identifying the largest width empty rectangular annulus amidst a set \(P\) of \(n\) points.

We define the problem in Section 2. Section 3 identifies the maximum width axis-parallel empty rectangular annulus.

2 Problem definition

Let \(P = \{p_1, p_2, \ldots, p_n\}\) be set of \(n\) points lying in the euclidean plane. \(CH(P)\) denotes the convex hull of the point set \(P\). \(d(p_i, p_j)\) denotes the distance between points \(p_i\) and \(p_j\). The perpendicular distance between \(p_i\) and a line or a line segment \(s\) is termed \(d(p_i, s)\).

A rectangle \(R\) of fixed orientation in the plane is said to be a minimum enclosing rectangle (MER) if (i) \(R\) contains all the members of \(P\) (ii) there does not exist another rectangle \(R'\) having area less than that of \(R\) and containing all points of \(P\). Observe that each of the four sides of a MER must contain at least one point of \(P\) and the points defining that MER are the points from \(CH(P)\). Two rectangles are said to be parallel if each side of one rectangle is parallel to a side of the other. \(bd(R)\) and \(int(R)\) denote the boundary and interior of a rectangle \(R\) respectively. Consider two mutually parallel rectangles \(R_1\) and \(R_2\) where \(R_2 \subset R_1\). The region \(A = R_1 \setminus (int(R_2) \cup bd(R_1))\) bounded by these two rectangles is said to be an empty rectangular annulus amidst a point set \(P\) if the unbounded region \(A\) does not contain any point of \(P\). Larger and the smaller rectangles of an annulus will be denoted as outer and inner rectangles respectively. From now on, the terms empty rectangular annulus and empty annulus will be used interchangeably. Now we define the width of an empty rectangular annulus among a set of points. Top-width of an empty annulus is defined as the perpendicular distance between the top sides of the outer and inner rectangles of that empty annulus. Similarly we can define right-width, bottom-width and left-width of an annulus. The maximum among the top-, right-, bottom- and left- width of an empty annulus is defined as the width of the empty annulus.

In this paper, our objective is to identify an axis-parallel empty annulus of the maximum or largest width. From now onwards, the term largest empty annulus is used to mean an axis-parallel empty annulus of maximum width (See Figure 2).

3 Axis-parallel annulus

To locate a largest empty annulus, we start with an axis-parallel MER \(R_1\) that encloses the point set \(P\). Observe that the MER \(R_1\) is a potential candidate for the outer rectangle that can generate an empty annulus. This observation motivates to design an iterative algorithm to recognize a largest empty annulus. In initial step of iteration, our objective is to find an axis-parallel inner rectangle \(R_2\) such that (i) \(R_2\) is wholly contained inside \(R_1\), (ii) the annulus formed by \(R_1\) and \(R_2\) does not contain any point of \(P\) and (iii) the width of an empty annulus thus formed is maximum. To compute the outer rectangle \(R_2\) for the given outer rectangle \(R_1\), the point set \(P\) is partitioned in the following way. We denote the top, right, bottom and left sides of \(R_1\) as \(t(R_1), r(R_1), b(R_1)\) and \(l(R_1)\) respectively. Analogously define the sides of the inner rectangle \(R_2\). Define \(T(R_2)\) as the set of points \(p_i\) inside the rectangle \(R_1\) such that each \(p_i \in T(R_1)\) has the top side of \(R_1\) as its nearest side among the other three sides of \(R_1\). Therefore \(T(R_1) = \{p_i : d(p_i, t(R_1)) \leq \min\{d(p_i, r(R_1)), d(p_i, b(R_1)), d(p_i, l(R_1))\}\). Similarly we can define the sets \(R(R_1), B(R_1)\) and \(L(R_1)\). Let \(p'_i\) be the element of \(T(R_1)\) such that \(d(p'_i, t(R_1)) = \min\{d(p_i, t(R_1)) : p_i \in T(R_1)\}\). Similarly define \(p'_i, p'_j\) and \(p'_i\) such that \(d(p'_i, r(R_1)) = \min\{d(p_i, r(R_1)) : p_i \in R(R_1)\}\), \(d(p'_j, b(R_0)) = \min\{d(p_i, b(R_1)) : p_i \in B(R_1)\}\) and \(d(p'_i, l(R_1)) = \min\{d(p_i, l(R_1)) : p_i \in L(R_1)\}\) respectively. We now have the following observation.

\[
R_1, R_2
\]

Figure 1: An axis-parallel largest empty annulus
Observation 1 For a given outer rectangle $R_1$, there exists a largest width empty annulus $A$ whose inner rectangle $R_2$ has the points $p'_t, p'_r, p'_b$ and $p'_l$ on its top, right, bottom and left sides respectively. 

Observe that, in initial as well as subsequent iterations, the boundary of the outer rectangle $R_1$ can contain at most four points of $P$ under general position assumption of the point set $P$. This implies, without loss of generality, assume that no two points of $P$ have the same $x$ or $y$ coordinates. Here we can find the following probable cases that may occur regarding the positions of above stated four points on the boundary of $R_1$.

1. Each side of $R_1$ contains a point of $P$.
2. Each corner of $R_1$ contains a point of $P$.
3. Each side of a pair of adjacent sides of $R_1$ contains a point of $P$ and the corner point of the remaining two sides of $R_1$ contains a point of $P$.
4. Each corner of a pair of opposite corners contains a point of $P$.

Observation 2 For a given point set $P$, we can discard at least one point $p$ from the boundary of $R_1$ for computing the outer rectangle of an empty annulus in the next iteration. This implies the outer rectangle (MER) for the next iteration is required to compute for the reduced point set $P \setminus \{p\}$ instead of $P$.

In the next iteration, we do the following steps.

(i) Remove at least one point $p$ from the boundary of the outer rectangle $R_1$ and compute the MER $R$ of the point set $P \setminus \{p\}$.

(ii) Select this MER as the current outer rectangle $R_1$.

(iii) For the given outer rectangle $R_1$, compute the inner rectangle $R_2$ by Observation 1.

Now we compute the number of iterative steps required to identify the desired axis-parallel empty annulus of maximum width. The Observation 2 implies that at least one points of $P$ can be discarded at each iteration. Therefore, the number of iterative steps required is at most $(n - 1)$. Thus we have the following theorem.

Theorem 1 Given a point set $P$ of $n$ points the Euclidean plane, we can identify a largest empty annulus among points of $P$ in $O(n^2)$ time and $O(n)$ space.

Proof: Computation of the MER as an outer rectangle of an empty annulus for a set of $n$ points requires linear time. For the given outer rectangle of an empty annulus, identifying the points on the boundary of the inner rectangle also requires $O(n)$ time. As stated earlier, the number of iterative steps is $O(n)$. Hence the theorem follows.

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References


